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CONTENTS

ABSTRACT

Shape representation has been a central issue in computer graphics and computer-aided geometric design. Many physical phenomena involve curves and surfaces that are monotone (in some directions) or are convex. The corresponding representation problem is given some monotone or convex data, and a monotone or convex interpolant is found. Standard interpolants need not be monotone or convex even though they may match monotone or convex data.

Several authors including Fritsch, Butland, McAllister, McLaughlin, Schumaker and Foley have investigated this problem. Most of their methods involve the utilization of quadratic splines or Hermite polynomials. In our investigation, we have adopted a similar approach. These methods require derivative information at the given data points. The key to the problem is the selection of the derivative values to be assigned to the given data points. Schemes for choosing derivatives were examined. Along the way, fitting given data points by a conic section has also been investigated as part of the effort to study shape-preserving quadratic splines.

I. INTRODUCTION

This project commenced in May 1987 and ended in October 1988, which includes a six-month, no-cost extension. Dr. Maria Lam was the Principal Investigator. She was assisted by a graduate student in the Applied Mathematics program at Hampton University.

Hampton University has benefited by this project. Through this grant, the Principal Investigator was able to obtain release time and physical resources to conduct research into the area that interests her. This project has been beneficial to the graduate program in Applied Mathematics by way of the research assistantship made available to the graduate student. The work done under this project will constitute part of the graduate student's dissertation. The Computer Science Department has also benefited. This project helped to fulfil part of the faculty research criterion set forth by the ACM Accreditation Board when the Department applied for accreditation in the spring of 1988. In short, this project has been beneficial to the University in several ways.

Two contributed papers were presented on research done under this project. The details of these are given in Sections II and III.

II. FITTING DATA BY CONIC SECTIONS

Data Identification:

The initial attempt has been to identify 2-D data assuming there is some underlying relation among the data points. Owing to the diversity of this problem, the study has focused on conic sections. A conic section can be represented either algebraically or geometrically /1,2/. Some of these representations require additional information than just the data points. In the absence of this information, in theory, a conic section can be represented algebraically by a quadratic relation

$$ax^2+2bxy+cy^2+2dx+2ey+f=0 \qquad \ldots\ldots (1)$$

or $PQP^T=0$ where

$$P=[x,y,1]$$

$$Q=\begin{bmatrix} a & b & d \\ b & c & e \\ d & e & f \end{bmatrix}.$$

Define angle s as

$$2s=\tan^{-1}(2b/(a-c))$$

or

$$s=\pi/4 \qquad \text{if } a=c.$$

Now effect a rotation of the coordinate system given by

$$x=X\cos(s) - Y\sin(s)$$

$$y=X\sin(s) + Y\cos(s).$$

Thus, equation (1) becomes

$$AX^2+CY^2+2DX+2EY+F=0 \qquad \ldots\ldots (2)$$

By examining the coefficiets of equation (2), the type of conic section (including the degenerate cases) can be determined /3/. Hence a conic section is completely determined by five data points since only five of the six coefficients in equation (1) are independent.

One can indirectly solve for these coefficients by the following method /4/. Assume the five data points are (x_i, y_i), (x_{i+1}, y_{i+1}), (x_{i+2}, y_{i+2}), (x_{i+3}, y_{i+3}) and (x_{i+4}, y_{i+4}). Form the 6x6 matrix M given by

$$M = \begin{bmatrix} x_i^2 & x_i y_i & y_i^2 & x_i & y_i & 1 \\ x_{i+1}^2 & x_{i+1} y_{i+1} & y_{i+1}^2 & x_{i+1} & y_{i+1} & 1 \\ x_{i+2}^2 & x_{i+2} y_{i+2} & y_{i+2}^2 & x_{i+2} & y_{i+2} & 1 \\ x_{i+3}^2 & x_{i+3} y_{i+3} & y_{i+3}^2 & x_{i+3} & y_{i+3} & 1 \\ x_{i+4}^2 & x_{i+4} y_{i+4} & y_{i+4}^2 & x_{i+4} & y_{i+4} & 1 \\ x^2 & xy & y^2 & x & y & 1 \end{bmatrix}$$

The equation

$$\det(M)=0 \qquad \ldots\ldots (3)$$

yields a quadratic equation in x, y. When $x=x_k$ and $y=y_k$ $(i<=k<=i+4)$, two of the rows of the matrix are identical, and equation (3) is satisfied. This implies that equation (3) represents the unique conic passing through these five points.

However, this algebraic method is found to be numerically unstable. The determination of the type of conic section depends largely on whether certain coefficients are positive, negative or zero. It is extremely sensitive to small perturbations in coefficients, which may alter the type of the conic section. When it is implemented in a computer program, the situation is even worse. Such method routinely involves considerable amount of algebraic operations. Their floating-point implementation on a computer almost invariably produces perturbations in the coefficients. The selection of reasonable tolerance is difficult. Computer algebras such as MACSYMA and SMP are ideal tools for this type of investigations because actual evaluation of expressions is delayed until the final step. This leads to improved accuracy, but, however it does not eliminate the problem.

Several sets of conic data are generated. For each data set, five consecutive data points are used for fitting. A rotation is performed to eliminate the mixed term. Then the ceofficients of the result are observed. This process is repeated until all points of a data set are exhausted.

Results:

The results of this study are mixed. As one would expect, it is difficult to identify these figures definitely.

To be identified as data that come from a circle, the generated coefficients A, C must be equal. This rarely happens. The best one can expect is to identify the data as elliptic. Our study shows that one set of circular data is identified as elliptic. A second set is identified as elliptic and hyperbolic with the elliptic outnumbering the hyherbolic. A third data set provides similar result. Elliptical data exhibit similar behavior.

Parabolas are identified by A=0 or C=0 but not both. For one set of parabolic data with the values of x several units apart, we are able to identify this trend every time. A second set of data, with closely spaced x values, does not fare well at all. It is correctly identified as parabola only on two occasions. For most part, it has been identified incorrectly as hyperbolic 37 times and elliptic 5 times. The third set is the most curious one. The first 18 fits indicate the data is hyperbolic, this is followed by 22 fits that indicate poarabolic data.

If the data is hyperbolic, the result of algebraic method leads one to conclude that the data is either hyperbolic or elliptic; however the former conclusion (i.e. hyperbolic) occurs more frequently.

In spite of this lack of positive identification, one pattern emerges. Parabolic data are identified as either parabolic or hyperbolic. It is the only type of data that will draw parabolic conclusion. Circles and ellipses are classified as ellipses or hyperbolas with the elliptical conclusion appearing more often than the other. This allows one to draw rough estimates on the type of conic section that the data could have originated from.

In summary, when given a set of conic data and algebraic method is used in attempt to identify it, one would assume the data is hyperbolic initially. If on some occasions the data is identified as parabolic and as hyperbolic on other occasions, one should accept the data as parabolic. If the data is identified as ellipse and hyperbola with ellipse more frequent than hyperbola, one should conclude it to be elliptic.

Data Fitting:

Our next attempt is to fit arbitrarily given data sets by conic section. Several data sets including some standard data sets to test shape-preserving interpolants are used. The procedure described in the previous subsection minus the rotation is repeated on these data sets. At the end of each fitting, the derivative at the point P(i+2) is calculated. These derivative values will be examined later.

When conic data is used, the result is rather consistent if consecutive data points are no more than a few units apart. The equations of the fits are approximately the same. The changes in coefficients are consistent and gradual. However, we are unable to recover the equation used to generate the data. These fits do not resemble the original equation whatsoever. For other data sets, if the five points used to do the fitting do not conform to conic shape, the fit will be poor if not downright impossible. This is usually indicated by some coefficients having values close to zero. The derivatives obtained from these conic fits generally preserve the monotone property of the data set. They also preserve the convexity of nice, well-behaved data sets, but fail under severe testing when this method is applied to some standard data sets used to test shape-preserving property of interpolants.

Data Used:

The following are some of the data sets used in this investigation.

Set 1

$(y-3)^2 = -20x$

x	-79	-76	-73	-72	-70	-66	-63	-62
y	42.75	41.99	41.21	41	40.42	39.33	38.5	38.21

Set 2

$x^2+y^2 = 100^2$

x	84	86.6	89.1	93.4	95.1	97.8	99.5
	100						
y	54	50	45.4	35.8	30.9	20.8	9.99
	0						

Set 3

$x^2+y^2 = 100^2$

x	84	85.4	86.6	88	89.1	90	92
	93.4	94	95.1	96	97	97.8	98.5
	99	99.5	100				
y	54	52	50	47.5	45.4	43.59	39.19
	35.8	34.12	30.9	28	24.31	20.8	17.25
	14.11	9.99	0				

Set 4

$y = x^3-x+1$

x	0	1	2	2.25	2.8	3	3.1
	3.6	4					
y	1	1	7	10.14	20.15	25	27.69
	44.06	61					

set 5
McAllister and Roulier data

x	0	2	4	6	8	10	12
y	0	2	44	88	132.1	1132.1	2132.2

Set 6
Roulier data

x	0	1	2	3	4	5	6
	7						
y	0	1.8	2.7	2.9	3	3	3.65
	4.7						

Set 7
Pruess data

x	22	22.5	22.6	22.7	22.8	22.9	23
	23.1	23.2	23.3	23.4	23.5	24	
y	523	543	550	557	565	575	590
	620	860	915	944	958	986	

Set 8
Akima data

x	0	2	3	5	6	8	9
	11	12	14	15			
y	10	10	10	10	10	10	10.5
	15	50	60	85			

Set 9
RNP 14 data (Fritsch and Carlson)

x	7.99	8.09	8.19	8.7
	9.2	10	12	15
	20			
y	0	0.0000276429	0.0437498	0.169183
	0.469428	0.94374	0.998636	0.999919
	0.999994			

III. SHAPE-PRESERVING INTERPOLATION

The problem is: Given points $x_1<x_2<...<x_n$ and values $\{y_i\}_{i=1}^n$, find f such that

$$f(x_i) = y_i \qquad i=1,2,...,n$$

and f preserves the monotonicity and/or convexity of the data. f is usually found by piecing f_i's together where f_i is defined over $[x_i,x_{i+1}]$ and

$$f_i(x_i) = y_i,\ f_i(x_{i+1}) = y_{i+1}.$$

Let $h_i=x_{i+1}-x_i$, $d_i=(y_{i+1}-y_i)/h_i$, $i=1,2,...,n-1$.

There are two general approaches to this problem. One employs geometric constructions which could be quite involved /5,6/. The other approach uses quadratic or cubic splines, together with carefully chosen derivative values or tension parameters /7,8,9/ to control the shape of the curve. Quadratic spline is chosen for this investigation because quadratic polynomials have one less extremum than cubic polynomials, hence better preserve the geometric properties of the given data. The derivatives of quadratics are linear, this makes them easier to study and manipulate. Hence we focus on a simple two-point Hermite interpolation problem involving quadratic polynomials: to find a quadratic polynomial h, if possible, such that

$$h(x_i)=y_i,\ h'(x_i)=s_i, \qquad i=1,2 \qquad ..... (4)$$

for some preassigned values $\{s_i\}_{i=1}^2$.

Conditions on the Derivatives:

Schumaker /6/ proved that there exists a quadratic polynomial solving this problem if and only if

$$\frac{s_1+s_2}{2} = \frac{y_2-y_1}{x_2-x_1} \qquad \ldots\ldots (5)$$

and explicitly defined this polynomial. Moreover, this polynomial is monotone on $I=[x_1,x_2]$ if $s_1s_2>=0$, it is convex on I if $s_1<s_2$, and concave if $s_1>s_2$.

When condition (5) fails, Schumaker showed that corresponding to every u in (x_1,x_2), it is possible to choose a functional value y^* and derivative s^* at u, and define a quadratic spline over I that satisfies condition (4). If $s_1s_2>=0$, then it is monotone on I if and only if $s_1s^*>=0$. If $s_1<s_2$, it is convex on I if and only if $s_1<=s^*<=s_2$. Similarly, if $s_1>s_2$, then it is concave if and only if $s_1>=s^*>=s_2$.

Selection of Derivatives:

The conditions stated in the previous subsection are deceptively simple. However, when both monotonicity and convexity are taken into consideration, the problam takes on a new dimension because one cannot satisfy one condition at the expense of the other. One generally approaches this problem by first assigning some carefully chosen derivative value to each data point, then modify some of these values if necessary to satisfy the monotone and/or convexity conditions, then the knots are inserted and their functional values and derivatives are determined, and finally the coefficients of the quadratic polynomials constituting the spline are calculated.

Schumaker /6/ used a weighted average of d_{i-1} and d_i as the value of s_i. McAllister and Roulier /5/ used a very involved geometric construction to obtain their shape-preserving quadratic splines. But their method can be identified as Schumaker's formulation with start-up derivative s_i defined as the harmonic mean of d_{i-1}, d_i and a modification scheme for s_i. DeVore and Yan /8/ improved upon their modification.

This investigation has focused on the selection of the values of s_i. The following method looks promising. The derivative s_i is defined as:

$$s_i = \begin{cases} \dfrac{d_{i-1}d_i}{td_{i-1}+(1-t)d_i} & \text{if } d_{i-1}d_i>0 \\ 0 & \text{otherwise} \end{cases}$$

where

$$t = \frac{1}{3}\left(1 + \frac{h_i}{h_{i-1}+h_i} \right).$$

This formula is symmetric in d_{i-1}, d_i because one can simplify $(1-t)$ to obtain

$$1-t = 1 - \frac{1}{3}\left(1 + \frac{h_i}{h_{i-1}+h_i} \right)$$

$$= \frac{1}{3} \left(2 - \frac{h_i}{h_{i-1}+h_i} \right)$$

$$= \frac{1}{3} \frac{h_i+2h_{i-1}}{h_{i-1}+h_i}$$

$$= \frac{1}{3} \left(1 + \frac{h_{i-1}}{h_{i-1}+h_i} \right).$$

This makes the formula independent of the order of the data points, i.e. the formula gives the same value when the data is processed from the left or from the right. This s_i also has the properties:

1. $\min(d_{i-1},d_i)<=s_i<=\max(d_{i-1},d_i)$

   thus the slope of the curve lies between the slopes of the two adjacent data segments,

2. $|s_i|<=\min(3|d_{i-1}|,3|d_i|)$ and

3. s_i reduces to the harmonic mean of d_{i-1}, d_i when x_{i-1}, x_i, x_{i+1} are equally spaced (i.e. when $h_{i-1}=h_i$).

All these properties make it very desirable.

The investigation was still in progress when this project came to an end. It will be carried on under the new project as grant NAG-1-948.

IV. CONTRIBUTED PAPERS

The following papers were presented on the research conducted under this project.

1. Shape Preserving Interpolants
   Maria H. Lam
   Annual Meeting of Virginia Academy of Science, Norfolk, Virginia, May 1987
   Virg. J. Sc., Vol. 38, No. 2, 71 (1987)

2. Fitting Data by a Conic Section
   Maria H. Lam
   Annual Meeting of Virginia Academy of Science, Charlottesville, Virginia, May 1988
   Virg. J. Sc., Vol. 39, No. 2, 103 (1988)

Abstracts of these papers are shown on following pages.

SHAPE PRESERVING INTERPOLANTS

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Recently there has been immense interest in shape-preserving interpolations. Most of these algorithms are quite involved. H. McLaughlin has proposed a simple, local method to generate interpolants which preserve the monotonicity and convexity of the given data. F. Fritsch and J. Butland have proposed a method to select values of derivatives at data points such that local monotone piecewise cubic interpolants will be obtained. These two methods are implmented by using computer algebra SMP. They are applied to several sets of data. Curves produced by these two methods are compared. Some of their properties are discussed.
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FITTING DATA BY A CONIC SECTION

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ABSTRACT

A conic section can be represented either algebraically or geometrically. In theory, it can be represented algebraically by a quadratic relation of the form

$$ax^2+2bxy+cy^2+2dx+2ey+f=0.$$

It is completely determined by five data points as only five of the six coefficients are independent. These coefficients can be determined by solving a system of five linear equations in six unknowns indirectly. This method is found to be numerically unstable. Its floating point implementation on a computer almost invariably produces perturbations in the coefficients which may alter the character of the conic section. Several sets of conic data are generated. For each data set, five consecutive data points are used for fitting. A rotation is performed to eliminate the mixed term. The coefficients of the result are observed and compared. This process is repeated until all points of a data set is exhausted. Result of this research will be presented.

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